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Time response functions and avalanche fluctuations in resistive plate chambers

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ABSTRACT

The time response function of RPCs is derived. First, primary electron distributions in the RPC gas gap are discussed. Then the exact expression for the fluctuations of an avalanche starting with a fixed number of primary electrons is derived, using Legler's model of avalanche multiplication in electronegative gases. By means of the Z-Transform formalism, the primary electron distributions and avalanche fluctuations are then combined and an analytic expression for the RPC time response function is derived. The solution is further used to discuss signal threshold and attachment effects. Finally, the time response function is evaluated for several primary ionization models.

1. Introduction

Several attempts have been made to derive analytic expressions for the key performance parameters, especially the time response function of an RPC. Although the basic physics processes that affect the time resolution are well understood and are easily formulated by Monte Carlo simulations, an analytic expression has the advantage of giving a deeper insight into the dependence and limits of performance parameters. The intrinsic time response function of an RPC was initially calculated for a single avalanche [1,2] then extended to a fixed number of several avalanches growing in parallel [3] and afterwards further extended to the general situation of several avalanches growing simultaneously with the number fluctuating according to a Poisson distribution [4]. The case of an electronegative mixture was considered explicitly in Ref. [5]. This latest model has still the drawback of being based on the un-physical assumption of a single electron per cluster, while in reality the cluster size distribution has approximate $1/n^2$ form.

In this report we derive an analytic expression for the RPC time response function that allows the inclusion of realistic cluster size distributions. By use of the Z-Transform and Laplace-Transform the final formula is expressed as a contour integral in the complex plane, which can be explicitly solved for some special cases and is well suited for numeric evaluation for the general case. This formulation allows a straight forward insight into effects of threshold and attachment. Analytic expressions for the average threshold crossing time and the r.m.s. time resolution are presented as well. After derivation of the general formula, the results from all the above mentioned reports are reproduced and finally, the results for realistic detector physics assumptions are presented.

Before proceeding to the main body of the paper we shall state the principle result: the time response function for an RPC using a gas with drift-velocity v, Townsend coefficient α, attachment coefficient η, assuming (1) an average number of n_0 efficient clusters which fluctuate according to a Poisson distribution (2) a cluster size distribution $f(m)$ with Z-Transform $F(z)$ having a radius of convergence r_F (3) avalanche multiplication according to Legler's avalanche model and (4) a threshold of n electrons, is given by

$$\rho(n,t) = \frac{1}{2\pi i}\oint \frac{e^{n_0F(z)}-1}{e^{n_0}-e^{n_0F(1/k)}}\frac{(1-k)^2nS}{(1-kz)^2} \times \exp\left[-St - n\frac{(1-k)(1-z)}{1-kz}e^{-St}\right]dz \quad (1)$$

with $S=(\alpha-\eta)v$ and $k=\eta/\alpha$. The integration is over a circle with radius $r_F<r<1/k$. Writing $z=r\exp(i\phi)$ and $dz=ir\exp(i\phi)\,d\phi$ and integrating over ϕ from $-\pi,\pi$ the expression is well suited for numerical evaluation.

This report is essentially the analytic formulation of the Monte Carlo simulations presented in Ref. [2], so the discussion on comparison between measurement and simulation is already presented there.

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2. Formulation of the problem

We want to find the RPC time response function $\rho(n,t)$ defined as the probability that the number of electrons in the gas gap crosses a threshold of n electrons between time t and $t+\mathrm{d}t$. If $g(m)$ is the probability that a charged particle leaves m primary electrons in the gas gap and $h(m,n,t)\,\mathrm{d}t$ is the probability that an avalanche starting with m primary electrons crosses a threshold of n electrons between time t and $t+\mathrm{d}t$, the time response function of the RPC is given by

$$\rho(n,t)=\frac{1}{a_0}\sum_{m=0}^{\infty}g(m)h(m,n,t) \tag{2}$$

where a_0 is a normalization constant. In case $h(m,n,t)$ were a normalized distribution, a_0 would be equal to unity. There is, however, a probability that all m primary electrons are attached and the avalanche never reaches the threshold, and the expression $\int h(m,n,t)\,\mathrm{d}t$ must give exactly this probability. Because in the presence of attachment $h(m,n,t)$ is not normalized to unity, the normalization factor a_0 will be different from 1. The above relation can be transformed into a relation between complex functions by use of the Z-transformation (see Appendix A): if $G(z)$ is the Z-Transform of $g(m)$ with radius of convergence r_G and $H(z,n,t)$ is the Z-Transform of $h(m,n,t)$ with radius of convergence r_H, using Eqs. (87), (89) and (90), Eq. (2) can be transformed into

$$\rho(n,t)=\lim_{z_2\to 1}(z_2-1)\times\left[\frac{z_2}{z_2-1}\frac{1}{2\pi\mathrm{i}}\frac{1}{a_0}\oint G(z)H\left(\frac{z_2}{z},n,t\right)\frac{\mathrm{d}z}{z}\right] \tag{3}$$

which results in the form

$$\rho(n,t)=\frac{1}{2\pi\mathrm{i}}\frac{1}{a_0}\oint G(z)H\left(\frac{1}{z},n,t\right)\frac{\mathrm{d}z}{z}. \tag{4}$$

The path of integration is a circle centered at $z=0$ with radius r satisfying $r_G<r<1/r_H$. This is the expression for the time response function that we will use throughout this report. The jth moment μ_j of $\rho(n,t)$ is given by

$$\begin{aligned}\mu_j&=\int t^j\rho(n,t)\,\mathrm{d}t\\&=\frac{1}{a_0}\sum_{m=0}^{\infty}g(m)\int t^j h(m,n,t)\,\mathrm{d}t\\&=\frac{1}{a_0}\sum_{m=0}^{\infty}g(m)u_j(m,n).\end{aligned} \tag{5}$$

In the Z-domain this reads as

$$\mu_j=\frac{1}{2\pi\mathrm{i}}\frac{1}{a_0}\oint G(z)U_j\left(\frac{1}{z},n\right)\frac{\mathrm{d}z}{z} \tag{6}$$

with $r_G<r<1/r_{U_j}$ and

$$\begin{aligned}U_j(z,n)&=\mathscr{Z}[u_j(m,n),m,z]\\&=\int t^j H(z,n,t)\,\mathrm{d}t.\end{aligned} \tag{7}$$

Because $\rho(n,t)$ must be normalized, i.e. $\mu_0:=1$ we have

$$a_0=\frac{1}{2\pi\mathrm{i}}\oint G(z)U_0\left(\frac{1}{z},n\right)\frac{\mathrm{d}z}{z}. \tag{8}$$

The average threshold crossing time $\bar{t}$ and the variance σ^2 of $\rho(n,t)$ are

$$\bar{t}=\mu_1,\quad \sigma^2=\mu_2-\mu_1^2. \tag{9}$$

The primary ionization distribution $g(m)$ follows from elementary assumptions of a Poissonian distribution of the primary clusters and a $1/n^2$ or more complicated cluster size distribution. For the avalanche distribution $h(m,n,t)$ we use Legler's model of avalanche fluctuations in electronegative gases [7].

3. Primary ionization, G(z)

Mechanisms and statistics of primary ionization caused by a charged particle traversing the gas gap are well known and were discussed e.g. in Ref. [2,8,9] for the used RPC gases. The probability $q(n)$ to find $n>0$ primary clusters in the gas gap follows to a good approximation a Poisson distribution [3], according to

$$q(0)=0,\quad q(n)=\frac{1}{\mathrm{e}^{n_0}-1}\frac{n_0^n}{n!},\quad n>0 \tag{10}$$

where n_0 is the average number of efficient clusters in the gas gap. We can estimate n_0 the following way: in case a primary electron is deposited very close to the 'anode' plate, the avalanche cannot grow sufficiently to cross the threshold. The charge induced by the electrons from an avalanche started with a single electron and growing with the average rate of $\exp(St)$ is [2,8,10]

$$Q(x)\approx\frac{E_w}{V_w}\frac{e_0}{\alpha-\eta}\,\mathrm{e}^{(\alpha-\eta)x}. \tag{11}$$

The efficient gap ends where this charge $Q(x)$ is just reaching the threshold Q_T before the electrons hit the resistive plate, so we have [2,8,10]

$$Q_T=Q(x),\quad x=\frac{1}{\alpha-\eta}\ln\left(\frac{Q_T}{e_0}\frac{\alpha-\eta}{E_w/V_w}\right). \tag{12}$$

Assuming the 300 μm single gap and four gap geometries (1a) and (1b) described in Ref. [2], we have $E_w\approx 1.5\,\mathrm{mm}^{-1}$. With typical values of $\alpha-\eta=110\,\mathrm{mm}^{-1}$ and $Q_T=20\,\mathrm{fC}$ we find $x=150\,\mu\mathrm{m}$. This means that the efficient gap is only $300-150=150\,\mu\mathrm{m}$, and with an average distance of 100 μm between clusters we find $n_0=1.5$ for a single gap. For a four gap RPC this gives $n_0=6$ efficient clusters.

The probability $f(n)$ to find n electrons per cluster, the so called cluster size distribution, follows approximately a $1/n^2$ law and the detailed distribution depends on the gas composition. The probability $g(n)$ to find n primary electrons in the gas gap is, therefore,

$$\begin{aligned}g(n)&=q(0)\delta_{0n}+q(1)f(n)\\&\quad+q(2)\sum_{m=0}^{n}f(n-m)f(m)\\&\quad+q(3)\sum_{m=0}^{n}\sum_{k=0}^{m}f(n-m)f(m-k)f(k)+\cdots.\end{aligned} \tag{13}$$

If $F(z)$ is the Z-Transform of $f(n)$ we use Eq. (88) to find the Z-Transform of $g(n)$, which results in

$$\begin{aligned}G(z)&=\frac{1}{\mathrm{e}^{n_0}-1}\left(0+\frac{n_0}{1!}F(z)+\frac{n_0^2}{2!}F(z)^2+\cdots\right)\\&=\frac{\mathrm{e}^{n_0F(z)}-1}{\mathrm{e}^{n_0}-1}.\end{aligned} \tag{14}$$

For the following discussion we use three cluster size distributions

$f(n)$	$F(z)$	r_F
δ_{1n}	$\frac{1}{z}$	0
$(1-q)\,q^{n-1}$	$\frac{1-q}{z-q}$	q
$\frac{6}{\pi^2}\frac{1}{n^2}$	$\frac{6}{\pi^2}\mathrm{Li}_2\left(\frac{1}{z}\right)$	1

(15)

The distributions are shown in Fig. 1 together with a more detailed prediction from HEED [6]. The first distribution in the above table represents a single electron per cluster and the second

one represents an exponential distribution with an average of $n_e = 1/(1-q)$ electrons per cluster. Both distributions are un-physical and are only used for illustration. The $1/n^2$ distribution reflects the fact that the cross-section for the energy transfer during an interaction between the incoming particle and a gas molecule is $1/E^2$ distributed, which is the basic assumption for the Landau distribution and serves as a minimum model respecting elementary physics. The function $\mathrm{Li}_2(x)$ is the dilogarithm function [11]. The inversion of $G(z)$ gives the number of primary electrons in the gas gap. With a single electron per cluster we have again the Poisson distribution for the number of primary electrons in the gas gap

$$g(0)=0, \quad g(n)=\frac{n_0^n}{n!(\mathrm{e}^{n_0}-1)}, \quad n>0. \tag{16}$$

For the exponential distribution we invert $G(z)$ by expanding it in a Laurent series around $z-q$ (Eq. (83)) and find the coefficient of $(z-q)^{-1}$, which results in $g(0)=0$ and for $n>0$ we have

$$\begin{aligned} g(n) &= \frac{n_0(1-q)q^{n-1}}{\mathrm{e}^{n_0}-1}\sum_{j=0}^{n-1}\binom{n-1}{j}\frac{1}{(j+1)!}\left(\frac{n_0(q-1)}{q}\right)^j \\ &= \frac{n_0(1-q)q^{n-1}}{\mathrm{e}^{n_0}-1}\,{}_1F_1\left(1-n,2,\frac{n_0(q-1)}{q}\right). \end{aligned} \tag{17}$$

The function ${}_1F_1(a,b;z)$ is the confluent hypergeometric function [13]. The inversion of $G(z)$ for the $1/n^2$ distribution is done by using Eq. (85) and $\mathrm{dLi}_2(z)/\mathrm{d}z = -\ln(1-z)/z$, which leads to

$$g(0)=0, \quad g(n)=\frac{a_n}{n!(\mathrm{e}^{n_0}-1)}, \quad n>0. \tag{18}$$

The a_n are defined by the recursive relation

$$a_n = \frac{6n_0(n-1)!}{\pi^2}\sum_{j=0}^{n-1}\frac{a_j}{j!(n-j)}, \quad n>0 \tag{19}$$

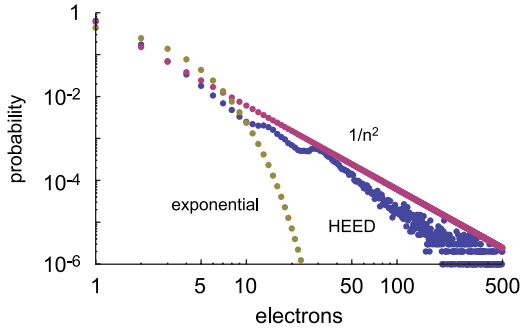

Fig. 1. Cluster size distribution for the exponential model, the $1/n^2$ model and a HEED calculation.

with $a_0=1$. The first two terms are

$$a_1=\frac{6n_0}{\pi^2}, \quad a_2=\frac{3n_0(12n_0+\pi^2)}{\pi^4} \quad \cdots. \tag{20}$$

An approximation for large n can be derived by using the trick from Appendix C, which gives

$$g(n)\approx\frac{\mathrm{e}^{n_0}}{\mathrm{e}^{n_0}-1}\left(\frac{6n_0}{\pi^2}\frac{1}{n^2}+\frac{72n_0^2}{\pi^4}\frac{\ln n}{n^3}+O\left(\frac{1}{n^4}\right)\right). \tag{21}$$

We see that the distribution behaves like $1/n^2$ for large n which is the important characteristic of the Landau distribution. In Fig. 2, the primary electron distribution for the exponential, the $1/n^2$ and the HEED cluster size distribution are shown.

4. Avalanche multiplication, $H(z,n,t)$

In this section we derive $H(z,n,t)$ for the exact solution and the continuous approximation of Legler's avalanche model.

4.1. Exact solution of Legler's avalanche model

Legler's avalanche model for electronegative gases [7] is based on the assumption that $\alpha\,\mathrm{d}x$ is the probability that an electron duplicates within a distance $\mathrm{d}x$ and $\eta\,\mathrm{d}x$ is the probability that an electron is attached within a distance $\mathrm{d}x$, independent of the electron's history. This defines the probability distribution $p(n,x)\,\mathrm{d}x$ to find n electrons between distance x and $x+\mathrm{d}x$. While Legler establishes an equation for $p(n,x)$ which contains products of $p(n,x)$ which he later transforms into the nonlinear Riccati differential equation [7], in Ref. [2] the following partial differential/difference equation is derived:

$$\begin{aligned} \frac{\partial p(n,x)}{\partial x} = &-p(n,x)n(\alpha+\eta)+p(n-1,x)(n-1)\alpha \\ &+p(n+1,x)(n+1)\eta. \end{aligned} \tag{22}$$

Assuming that the avalanche starts with m electrons at $x=0$ adds the condition $p(n,0)=\delta_{mn}$. The equation can be solved by first performing the Z-Transform $P(z,x)=\mathscr{Z}[p(n,x),n,z]$ and using Eq. (92), which gives

$$\frac{\partial P(z,x)}{\partial x}=\frac{\partial P(z,x)}{\partial z}((\alpha+\eta)z-\alpha-\eta z^2) \tag{23}$$

with the condition $P(z,0)=1/z^m$. The solution of this equation is derived in Appendix E. In the following we write $p(m,n,x)$ for the probability that an avalanche starting with m electrons contains n electrons at x. Defining $k=\eta/\alpha$ and the effective Townsend coefficient $\alpha_e=\alpha-\eta=\alpha(1-k)$, the solution from Eq. (23)

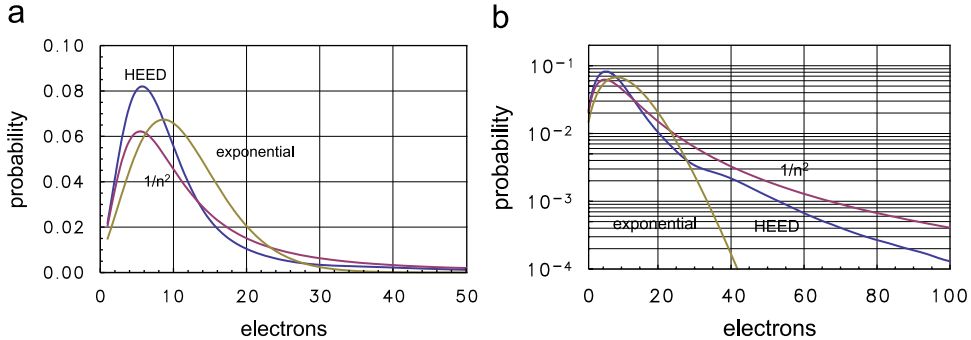

Fig. 2. Primary electron distribution with $n_0=5$ for the exponential cluster size distribution with $n_e=2.27$, the $1/n^2$ distribution and the HEED cluster size distribution. (a) linear scale, (b) logarithmic scale. All distributions have the median at around $n=10$ electrons.

reads as

$$P(m,z,x)=\left(\frac{k(z-1)\mathrm{e}^{\alpha_e x}-kz+1}{(z-1)\mathrm{e}^{\alpha_e x}-kz+1}\right)^m. \tag{24}$$

To check the normalization of $p(m,n,x)$ we use Eq. (91) and verify that

$$\sum_{n=0}^{\infty}p(m,n,x)=P(m,1,x)=1. \tag{25}$$

The average number of electrons at position x is calculated by use of Eqs. (87), (90) and (92), which gives

$$\begin{aligned}\overline{n}(m,x)&=\sum_{n=0}^{\infty}np(m,n,x)\\&=\lim_{z\to 1}\left(-z^2\frac{\mathrm{d}P(m,z,x)}{\mathrm{d}z}\right)\\&=m\,\mathrm{e}^{\alpha_e x}.\end{aligned} \tag{26}$$

For finding the variance σ^2 we calculate

$$\begin{aligned}\overline{n^2}&=\sum_{n=0}^{\infty}n^2p(m,n,x)\\&=\lim_{z\to 1}z^2\frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}P(m,z,x)}{\mathrm{d}z}\right)\\&=\frac{1+k}{1-k}m\,\mathrm{e}^{\alpha_e x}(\mathrm{e}^{\alpha_e x}-1)+m^2(\mathrm{e}^{\alpha_e x})^2\end{aligned} \tag{27}$$

and get

$$\sigma^2=\overline{n^2}-\overline{n}^2=\frac{1+k}{1-k}m\,\mathrm{e}^{\alpha_e x}(\mathrm{e}^{\alpha_e x}-1). \tag{28}$$

In the following we write the average of a single electron avalanche as $\overline{n}:=\overline{n}(1,x)$. The probability $p(m,n,x)$ is given by the inverse Z-Transform of $P(m,z,x)$. The function has an mth order pole at $z_0=(\overline{n}-1)/(\overline{n}-k)$ and by using Eq. (98) we can evaluate the integral of Eq. (83). Defining

$$a=\frac{1-k}{\overline{n}-k},\quad b=1-a=\frac{\overline{n}-1}{\overline{n}-k} \tag{29}$$

we get $p(0,n,x)=\delta_{n0}$ and for $m>0$ we have

$$p(m,0,x)=(kb)^m \tag{30}$$

$$\begin{aligned}p(m,n>0,x)&=(\overline{n}a^2)^m b^{(n-m)}\\&\quad\times\sum_{j=0}^{m-1}\binom{n-1}{m-j-1}\binom{m}{j}\left(\frac{kb^2}{\overline{n}a^2}\right)^j.\end{aligned} \tag{31}$$

For $m=1$, i.e. for an avalanche starting with a single electron, we have [7]

$$\begin{aligned}p(1,0,x)&=k\frac{\overline{n}-1}{\overline{n}-k}\\p(1,n>0,x)&=\overline{n}\left(\frac{1-k}{\overline{n}-k}\right)^2\left(1-\frac{1-k}{\overline{n}-k}\right)^{n-1}.\end{aligned} \tag{32}$$

Using the hypergeometric function ${}_2F_1(u,v,w;z)$ [12], Eq. (31) can be written as

$$\begin{aligned}p(m,n\geqslant m,x)&=(\overline{n}a^2)^m b^{(n-m)}\frac{m}{n}\binom{n}{m}\\&\quad\times{}_2F_1\left(1-m,-m,n-m+1;\frac{kb^2}{\overline{n}a^2}\right)\end{aligned} \tag{33}$$

$$\begin{aligned}p(m,n<m,x)&=(\overline{n}a^2)^n b^{(m-n)}\binom{m}{n}\\&\quad\times{}_2F_1\left(-n,1-n,m-n+1;\frac{kb^2}{\overline{n}a^2}\right).\end{aligned} \tag{34}$$

This is the exact expression for the probability to find n electrons at distance x for an avalanche that has started with m electrons at $x=0$, in a gas with attachment coefficient k. Without using the Z-Transform the expression can be calculated by m-times self convoluting Eq. (32).

4.2. Continuous approximation of Legler's avalanche model

For large numbers of $\overline{n}$ the distribution $p(m,n,t)$ can be assumed to be continuous in n and we can approximate it. The Z-Transform $P(m,z,t)$ becomes the Laplace Transform of the continuous approximation $Q(m,s,t)$ if we replace z by e^s and approximate for small s (Appendix C)

$$\begin{aligned}Q(m,s,x)&=\left(\frac{k(\mathrm{e}^s-1)\overline{n}-\mathrm{e}^s k+1}{(\mathrm{e}^s-1)\overline{n}-\mathrm{e}^s k+1}\right)^m\Bigg|_{s\to 0}\\&\approx\left(\frac{\overline{n}ks+1-k}{\overline{n}s+1-k}\right)^m.\end{aligned} \tag{35}$$

The probability $q(m,n,x)$ that an avalanche starting with m electrons at $x=0$ contains n electrons at x is then given by the inverse Laplace Transform of this expression.

$$\begin{aligned}q(m,n,x)&=\mathscr{L}^{-1}[Q(m,s,x),s,n]\\&=k^m\delta(n)+\frac{(1-k)^2}{\overline{n}}\exp\left(-n\frac{(1-k)}{\overline{n}}\right)\\&\quad\times k^{m-1}\sum_{j=0}^{m-1}\binom{m}{j+1}\frac{1}{j!}\left(\frac{n(1-k)^2}{\overline{n}k}\right)^j\\&=k^m\delta(n)+\frac{(1-k)^2}{\overline{n}}m\exp\left(-n\frac{(1-k)}{\overline{n}}\right)\\&\quad\times k^{m-1}{}_1F_1\left(1-m,2,-\frac{n(1-k)^2}{k\overline{n}}\right).\end{aligned} \tag{36}$$

This is the continuous approximation of Eqs. (30) and (31). For a single primary electron this becomes

$$q(1,n,x)=k\delta(n)+\frac{(1-k)^2}{\overline{n}}\exp\left(-n\frac{(1-k)}{\overline{n}}\right)$$

in lieu of Eq. (32). For $k=0$ we have

$$q(m,n,x)=\frac{1}{(m-1)!}\left(\frac{n}{\overline{n}}\right)^{m-1}\frac{1}{\overline{n}}\exp\left(-\frac{n}{\overline{n}}\right). \tag{37}$$

4.3. Exact expression for $H(z,n,t)$

If the avalanche proceeds with velocity v, we have $x=vt$, defining the time development of the avalanche. Since $\alpha np(m,n,x)\,\mathrm{d}x$ is the probability that an avalanche starting with m electrons contains n electrons at x and $n+1$ electrons at $x+\mathrm{d}x$, the expression $\alpha vnp(m,n,vt)\,\mathrm{d}t$ is the probability that the avalanche crosses a threshold of n electrons between time t and $t+\mathrm{d}t$. On the other hand, in case of attachment, there is also a probability that the avalanche returns below the threshold before again crossing it in positive direction. The question arises which of the threshold crossings one uses, and the most reasonable assumption is the use of the 'averaged' avalanche growth with the effective Townsend coefficient $\alpha-\eta$ at this point. The probability that an avalanche starting with m electrons crosses a threshold of n electrons between time t and $t+\mathrm{d}t$ is, therefore,

$$\begin{aligned}h(m,n,t)&=a_{mn}(\alpha-\eta)vnp(m,n,vt)\\&=a_{mn}Snp(m,n,vt)\end{aligned} \tag{38}$$

where we have defined

$$S=(\alpha-\eta)v. \tag{39}$$

The constants a_{mn} must be chosen such that $\int h(m,n,t)\,\mathrm{d}t$ is equal to the probability that the avalanche crosses the threshold at all. In order to determine a_{mn} we must first calculate $c_{nm} = S\int np(m,n,t)\,\mathrm{d}t$. For this we calculate the Z-Transform of $p(m,n,t)$ with respect to m and n

$$\begin{aligned} P_{AB}(z_1,z_2,t) &= \mathscr{Z}[p(m,n,t),m,z_1,n,z_2] \\ &= \sum_{m=0}^{\infty}\frac{P(m,z_2,t)}{z_1^m} \\ &= \frac{z_1[(kz_2-1)-\overline{n}(z_2-1)]}{\overline{n}(k-z_1)(z_2-1)+(z_1-1)(kz_2-1)} \end{aligned} \tag{40}$$

with $\overline{n} = \exp(St)$. The Z-Transform of c_{mn} is, therefore,

$$\begin{aligned} C(z_1,z_2) &= \mathscr{Z}\left[\int_0^{\infty} Snp(m,n,t)\,\mathrm{d}t,m,z_1,n,z_2\right] \\ &= S\int_0^{\infty} -z_2\frac{\partial P_{AB}(z_1,z_2,t)}{\partial z_2}\,\mathrm{d}t \\ &= \frac{(1-k)z_1z_2}{(z_2-1)(z_1-k)(z_1z_2-1)}. \end{aligned}$$

The c_{nm} are given by the inverse Z-Transform of $C(z_1,z_2)$

$$\begin{aligned} c_{mn} &= 1-k^m, \quad n\geqslant m \\ &= k^{m-n}(1-k^n), \quad n<m. \end{aligned} \tag{41}$$

We can interpret these expressions in the following way. If $n>m$, the number of electrons crosses a threshold of n if at least one of the m primary electrons is not attached, which has a probability of $1-k^m$. If $n<m$, i.e., the threshold n is smaller than the number m of primary electrons, $m-n$ electrons must be attached in order to arrive at the threshold level (probability k^{m-n}), and then at least one of the remaining n electrons must not be attached and develop and avalanche (probability $1-k^n$). We therefore conclude that in Eq. (38) the constant $a_{mn}=1$ and $h(m,n,t)$ is already properly normalized.

Finally, we can calculate $H(z,n,t)$. The Z-Transform of $p(m,n,x)$ with respect to m is given by inverting P_{AB}. The inverse Z-Transform with respect to z_2 is then given by

$$H(z,n,t) = Sn\frac{1}{2\pi\mathrm{i}}\oint P_{AB}(z,z_2,t)z_2^{n-1}\,\mathrm{d}z_2. \tag{42}$$

The integrand has a first order pole at $z_2 = (\overline{n}(k-z_1)+z_1-1)/(\overline{n}(k-z_1)+k(z_1-1))$, so we can evaluate the integral by use of Eq. (98) and get the final expression

$$\begin{aligned} H(z,n,t) &= \frac{Snz\overline{n}(1-k)^2}{[k(\overline{n}-1)-z(\overline{n}-k)]^2} \\ &\quad\times\left[1-\frac{(1-k)(1-z)}{k(\overline{n}-1)-z(\overline{n}-k)}\right]^{n-1} \end{aligned} \tag{43}$$

with $r_H = k(\overline{n}-1)(\overline{n}-k)$. The function $U_0(z,n)$ for calculating the normalization is given by

$$\begin{aligned} 1/zU_0(1/z,n) &= \int_0^{\infty} 1/zH(1/z,n,t)\,\mathrm{d}t \\ &= \frac{(1-k)(1-z^n)}{(1-z)(1-kz)}, \quad r_{U_0}=1. \end{aligned}$$

The analytic expressions $U_j(z,n)$ for calculation of the higher moments are too complicated to be useful and the average and standard deviation must be calculated by numerical integration of $\rho(n,t)$.

4.4. Continuous approximation of $H(z,n,t)$

For the continuous approximation $q(m,n,t)$, the probability for an avalanche starting with m electrons to cross a threshold of n electrons between time t and $t+\mathrm{d}t$ is, like in the previous section, given by

$$h(m,n,t) = Snq(m,n,vt). \tag{44}$$

We verify by the same methods the normalization of $h(m,n,t)$, giving

$$\int_{-\infty}^{\infty} h(m,n,t)\,\mathrm{d}t = 1-k^m. \tag{45}$$

To find $H(z,n,t)$ we proceed as before

$$\begin{aligned} Q_{AB}(z,s,x) &= \sum_{m=0}^{\infty}\frac{Q(m,s,x)}{z^m} \\ &= \frac{z(s\overline{n}+1-k)}{s\overline{n}(z-k)+(z-1)(1-k)} \end{aligned} \tag{46}$$

$$\begin{aligned} H(z,n,t) &= nS\mathscr{L}^{-1}[Q_{AB}(z,s,x),s,n] \\ &= \frac{z(1-k)^2}{(z-k)^2}\frac{nS}{\overline{n}}\exp\left(-n\frac{(1-k)(z-1)}{z-k}\frac{1}{\overline{n}}\right) \end{aligned} \tag{47}$$

with $r_H = k$. The functions $U_0(z,n)$, $U_1(z,n)$ and $U_2(z,n)$ (Eq. (7)) for calculation of the normalization, the average threshold crossing time and the standard deviation are given by

$$\frac{1}{z}U_0(1/z,n) = J(z) \tag{48}$$

$$\frac{1}{z}U_1(1/z,n) = \frac{J(z)}{S}(\gamma+\ln n+\ln M(z)) \tag{49}$$

$$\begin{aligned} \frac{1}{z}U_2(1/z,n) &= \frac{J(z)}{S^2}[\gamma^2+\pi^2/6+(2\gamma+\ln n)\ln n \\ &\quad+2(\gamma+\ln n)\ln M(z)+\ln^2 M(z)] \end{aligned} \tag{50}$$

with $r_{U_0} = r_{U_1} = r_{U_2} = 1$. The constant γ is the Euler–Mascheroni constant and we have defined

$$J(z) = \frac{1-k}{(1-kz)(1-z)}, \quad M(z) = \frac{(1-k)(1-z)}{1-kz}. \tag{51}$$

5. Time response function, $\rho(n,t)$

We have now all the ingredients to write down the RPC time response function $\rho(n,t)$. Before calculating $\rho(n,t)$ explicitly for different primary ionization models we derive a few general properties that are independent of the specific form of the primary electron distribution $G(z)$.

5.1. Exact solution

The time response function using the exact solution of Legler's avalanche model is given by using $H(z,n,t)$ from Eq. (43) and reads as

$$\begin{aligned} \rho(n,t) &= \frac{1}{a_0}\frac{1}{2\pi\mathrm{i}}\oint G(z)\frac{nS\,\mathrm{e}^{-St}(1-k)^2}{[(1-kz)-k\,\mathrm{e}^{-St}(1-z)]^2} \\ &\quad\times\left[1-\frac{\mathrm{e}^{-St}(1-k)(1-z)}{(1-kz)-k\,\mathrm{e}^{-St}(1-z)}\right]^{n-1}\mathrm{d}z \end{aligned}$$

where the path of integration is a circle centered at $z=0$ with radius r where

$$r_G < r < \frac{1}{k}\frac{\mathrm{e}^{St}-k}{\mathrm{e}^{St}-1}. \tag{52}$$

The normalization constant a_0 can be calculated by

$$a_0 = \frac{1}{2\pi\mathrm{i}}\oint G(z)\frac{(1-k)(1-z^n)}{(1-z)(1-kz)}\,\mathrm{d}z \tag{53}$$

or directly by Eq. (5) with Eq. (41)

$$\begin{aligned} a_0 &= \sum_{m=0}^{\infty} g(m) \int h(m,n,t)\,\mathrm{d}t \\ &= \sum_{m=0}^{n} g(m)(1-k^m) \\ &\quad + \sum_{m=n+1}^{\infty} g(m)k^{m-n}(1-k^n) \\ &= \left(\frac{1}{k^n}-1\right)G(1/k) + \sum_{m=0}^{n} g(m)\left(1-\frac{k^m}{k^n}\right). \end{aligned} \tag{54}$$

5.2. Continuous approximation

As above we calculate a_0 by Eqs. (5) and (45) and get

$$a_0 = \sum_{m=0}^{\infty} g(m)(1-k^m) = 1 - G(1/k). \tag{55}$$

We can also find the same expression by performing the large n limit of Eq. (54). The time response function for the continuous avalanche approximation Eq. (47) is, therefore,

$$\begin{aligned} \rho(n,t) = \frac{1}{2\pi \mathrm{i}} \oint \frac{G(z)}{1-G(1/k)} \frac{nS(1-k)^2}{(1-kz)^2} \\ \times \exp\left[-St - n\frac{(1-k)(1-z)}{1-kz}\,\mathrm{e}^{-St}\right] \mathrm{d}z. \end{aligned} \tag{56}$$

The path of integration is a circle of radius r with $r_G < r < 1/k$ centered at $z = 0$. We see that Eq. (56) is a good approximation of Eq. (52) if $\exp(St) \gg 1$, because then we have $(1-x)^{n-1} = \exp[(n-1)\ln(1-x)] \approx \exp(-nx)$ for $x \ll 1$. This time response function has several interesting properties which we investigate in the following.

Threshold independence: First we observe that by scaling the threshold n by a factor $\beta = \mathrm{e}^{\ln\beta}$, the distribution is simply shifted in time by $T = -(\ln\beta)/S$

$$\begin{aligned} \rho(\beta n,t) &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \frac{G(z)J(z)M(z)}{1-G(1/k)} Sn \\ &\quad \times \exp[-St + \ln\beta - nM(z)\,\mathrm{e}^{-St+\ln\beta}]\,\mathrm{d}z \\ &= \rho(n, t-(\ln\beta)/S) \end{aligned}$$

where $J(z)$ and $M(z)$ are from Eq. (51). We learn that the shape of the time response function, and therefore the time resolution of an RPC, is independent of the applied threshold in case the requirement $\exp(St) \gg 1$ is fulfilled, i.e., if the number of electrons is the gap is large.

Cluster transformation: In absence of attachment ($k = 0$) we have

$$\rho(n,t) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} G(z) Sn \exp[-St - n(1-z)\,\mathrm{e}^{-St}]\,\mathrm{d}z \tag{57}$$

where Γ denotes the path of integration, namely a circle centered at the origin with a radius of $r_G < r < \infty$. By using the variable transformation

$$z = \frac{y-a}{1-a} \to y = (1-a)z + a, \quad \mathrm{d}z = \frac{\mathrm{d}y}{1-a}$$

the integral transforms into

$$\begin{aligned} \rho(n,t) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma'} G\left(\frac{y-a}{1-a}\right) S \frac{n}{1-a} \\ \times \exp\left[-St - \frac{n}{1-a}(1-y)\,\mathrm{e}^{-St}\right] \mathrm{d}y. \end{aligned}$$

The path of integration Γ' is a circle centered at $y = a$ with radius r' where $(1-a)r_G < r' < (1-a)/k$. Therefore, the time response function for a primary electron distribution $G(z)$ and threshold n is equal to the time response function for a primary electron distribution $\bar{G}(z(y)) = G((y-a)/(1-a))$ with a threshold of $n/(1-a)$. Because $z(1) = 1$ we know that $\bar{G}(z(1)) = 1$ which means that it the Z-Transform of a normalized probability distribution. Since the variance σ is independent of the threshold, the time resolution for a cluster size distribution $G(z)$ and $G(z(y))$ is equivalent. As an example we assume a single primary electron and, therefore, $G(z) = 1/z$. We have $G(z(y)) = (1-a)/(y-a)$ and, therefore,

$$\mathscr{Z}^{-1}[G(z(y)), y, n] = (1-a)a^{n-1} \tag{58}$$

which represents an exponential electron distribution. The time resolution for a single primary electron and an exponential primary electron distribution is therefore equivalent.

Attachment transformation: By using a variable transformation

$$\frac{1-z}{1-kz} = 1 - y \to z(y) = \frac{y}{1-k+ky} \tag{59}$$

$$\mathrm{d}z = \frac{(1-k)}{(1+k(y-1))^2}\,\mathrm{d}y \tag{60}$$

the time response function from Eq. (56) transforms into

$$\begin{aligned} \rho(n,t) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma''} \frac{G(z(y))}{1-G(1/k)} \\ \times Sn' \exp[-St - n'(1-y)\,\mathrm{e}^{-St}]\,\mathrm{d}y \end{aligned} \tag{61}$$

where $n' = (1-k)n$. This means that the time response function for a primary electron distribution $G(z)$ with attachment k and threshold n is equivalent to the time response function for a primary electron distribution $G(z(y))$ without attachment and the threshold $n' = (1-k)n$. Because $z(1) = 1$ the distribution $G(z(y))$ is the Z-Transform of a normalized probability distribution. To find the path of integration Γ'' we note that the above transformation is a so called bilinear transformation which has the general form $y = (a+bz)(c+\mathrm{d}z)$. It has the property that circles in the z-plane are transformed into circles in the y-plane. The above expression transforms a circle of radius r centered at $z = 0$ into a circle with it is center on the real axis and crossing this axis at

$$y_1 = \frac{-r(1-k)}{1+kr}, \quad y_2 = \frac{r(1-k)}{1-kr}. \tag{62}$$

The circle of radius $r = 1/k$, therefore, transforms into the entire half-plane 'right' of $y = -(1-k)/2k$.

Average and variance: The average threshold crossing time $\bar{t}$ and the variance σ of the time response function are given by Eq. (9) using Eqs. (6) and (7) and Eqs. (48)–(50). Defining

$$\begin{aligned} \kappa_1 &= \frac{1}{2\pi \mathrm{i}} \oint \frac{G(z)J(z)\ln M(z)}{1-G(1/k)}\,\mathrm{d}z \\ \kappa_2 &= \frac{1}{2\pi \mathrm{i}} \oint \frac{G(z)J(z)\ln^2 M(z)}{1-G(1/k)}\,\mathrm{d}z \end{aligned} \tag{63}$$

with $r_G < r < 1$, a lengthy but elementary calculations gives

$$S\bar{t} = (\ln n + \gamma + \kappa_1), \quad S^2\sigma^2 = \frac{\pi^2}{6} + \kappa_2 - \kappa_1^2. \tag{64}$$

As expected, a change of threshold n results in a simple shift of $\bar{t}$, while the variance σ^2 is independent of the threshold.

6. Time response function for *m* primary electrons

The time response function for a fixed number of m primary electrons can be directly written down using Eqs. (30), (31) and (38) for the exact avalanche solution, and Eqs. (36) and (44) for the continuous approximation. For the continuous avalanche

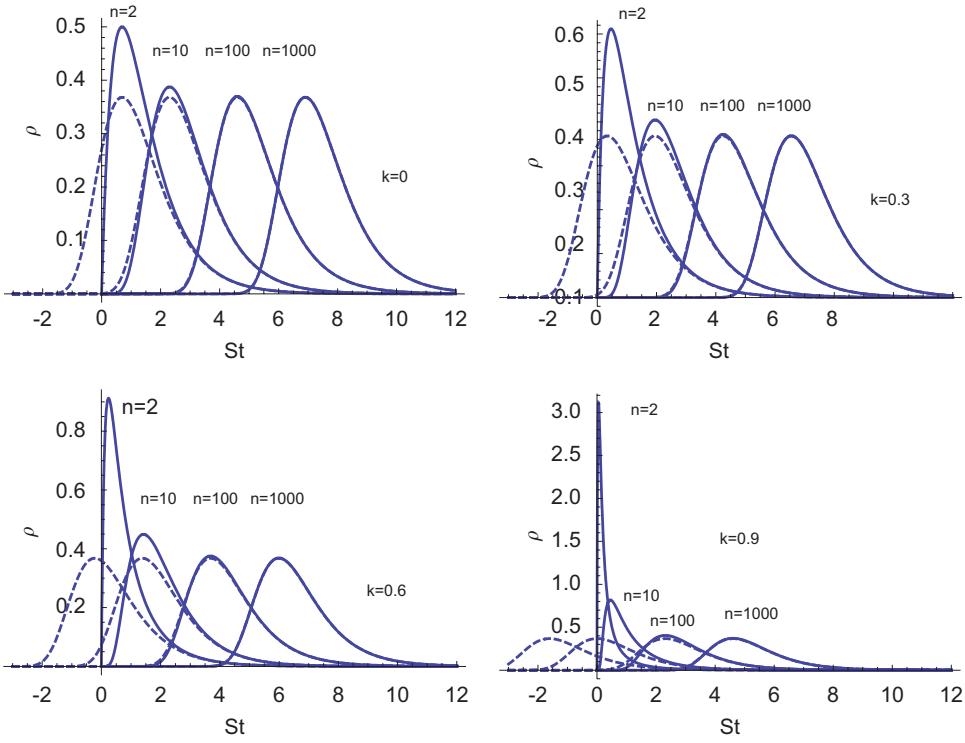

Fig. 3. Time response function for a single primary electron ($m=1$) for thresholds of $n=2,\ 10,\ 100,\ 1000$ electrons. The four figures correspond to the four different attachment coefficients $k=0,\ 0.3,\ 0.6,\ 0.9$. The solid line shows the exact solution while the dashed line shows the continuous approximation.

approximation this reads as

$$\rho(n,t)=\frac{nS}{a_0}(1-k)^2\exp(-St-n(1-k)\,\mathrm{e}^{-St})\times k^{m-1}\sum_{j=0}^{m-1}\binom{m}{j+1}\frac{1}{j!}\left(\frac{n(1-k)^2}{k}\,\mathrm{e}^{-St}\right)^j \tag{65}$$

where a_0 a normalization constant. Fig. 3 shows the time response function $\rho(n,1)$ for the exact avalanche model and the continuous approximation for a single primary electron ($m=1$), different thresholds n and different attachment coefficients k. We see that for thresholds $n\gg 1$, the shape of $\rho(n,t)$ for the exact avalanche model and the continuous approximation coincide. We also see that a change of threshold n results in a shift of ρ without changing the shape in case $n\gg 1$. In order to see how Eq. (65) is calculated from Eq. (56) we use $G(z)=1/z^m$ for the m primary electrons and we use the transformed Eq. (61). This results in

$$\begin{aligned}\rho(n,t)&=\frac{1}{2\pi\mathrm{i}}\frac{1}{1-k^m}\oint\left(k+\frac{1-k}{y}\right)^m Sn'\times\exp(-St-n'(1-y)\,\mathrm{e}^{-St})\,\mathrm{d}y\\&=\frac{n'S}{1-k^m}\exp(-St-n'\,\mathrm{e}^{-St})k^m\times\frac{1}{2\pi\mathrm{i}}\times\sum_{j=0}^{m}\binom{m}{j}\frac{(1-k)^j}{k^j}\oint\frac{1}{y^j}\exp(n'y\,\mathrm{e}^{-St})\,\mathrm{d}y.\end{aligned} \tag{66}$$

The integral evaluates to zero for $j=0$ and for $j>0$, by use of Eq. (98), we get

$$\frac{1}{2\pi\mathrm{i}}\oint\frac{1}{y^j}\exp(n'y\,\mathrm{e}^{-St})\,\mathrm{d}y=\frac{1}{(j-1)!}(n'\,\mathrm{e}^{-St})^{j-1}.$$

This above expression is equal to Eq. (65) with the normalization factor being $a_0=1-k^m$. In absence of attachment ($k=0$) we have

$$\rho(n,t)=\frac{Sn^m}{(m-1)!}\exp(-mSt-n\,\mathrm{e}^{-St}). \tag{67}$$

This distribution has a peak at $St=\ln n/m$. The peak has the value $S(m/e)^m/(m-1)!$. Curiously the distribution for $(m+1)$ crosses exactly through this peak as seen in Fig. 4. In the same figure we see the time response function for different attachment coefficients, and we observe that for larger attachment, meaning $k\to 1$, the curves for different primary electron numbers are very similar. Mathematically this is seen by the fact that for $1-k\ll 1$ the factor $(1-k)^{2j}$ in Eq. (65) decreases very quickly with j such that the $j=0$ term of the sum dominates, giving

$$\rho(n,t)\to nS(1-k)\exp(-St-n(1-k)\,\mathrm{e}^{-St}).$$

The time response function, therefore, becomes equal to the one for a single primary electron with an effective threshold of $n'=(1-k)n$.

Next we calculate the average threshold crossing time and the time resolution for a fixed number of m primary electrons. κ_1 from Eq. (63) becomes

$$\kappa_1(m)=\frac{1}{2\pi\mathrm{i}}\oint\frac{1}{z^m}\frac{J(z)\ln M(z)}{1-k^m}\,\mathrm{d}z. \tag{68}$$

The expression has an mth order pole at $z=0$ and we can evaluate the integral using Eq. (98)

$$\bar{St}(m)=\ln n+\gamma+\ln(1-k)-\frac{1}{1-k^m}\times\sum_{j=1}^{m-1}\frac{1}{j}[(1+k^m)-(k^j+k^{m-j})]. \tag{69}$$

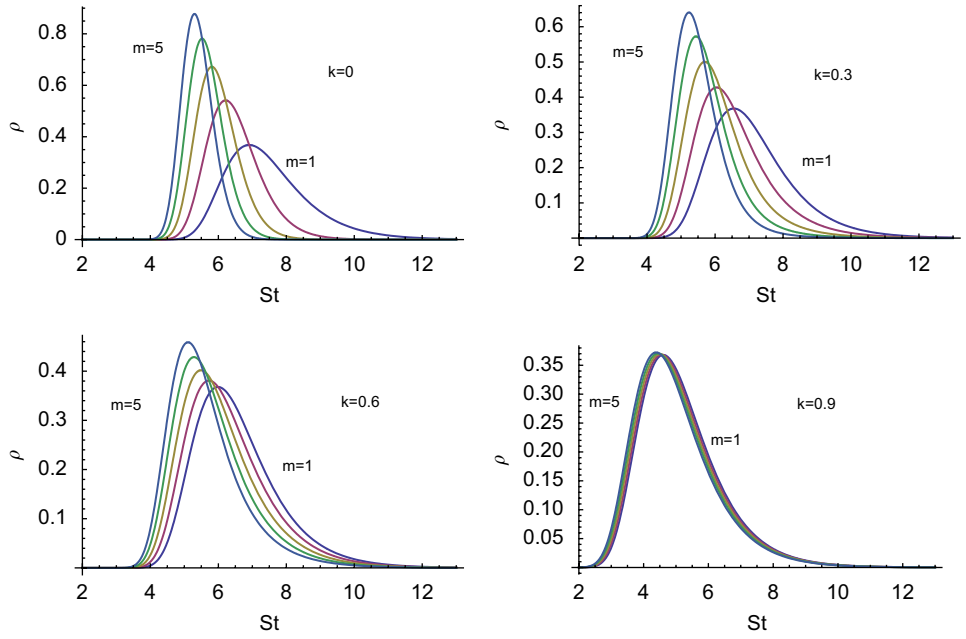

Fig. 4. Time response function for a fixed number of $m = 1,\ 2,\ 3,\ 4,\ 5$ primary electrons and threshold $n = 1000$. The four figures correspond to $k = 0,\ 0.3,\ 0.6,\ 0.9$.

In the same way we find κ_2 and get for the variance σ^2 (Eq. (64)) the lengthy but elementary expression

$$\begin{aligned} S^2\sigma(m)^2 = {} & \frac{\pi^2}{6} + \frac{2}{1-k^m} \\ & \times \sum_{j=1}^{m-1}\sum_{r=1}^{j-1}\left[\frac{1}{jr}(1-k^{m-j}+k^j-k^m)\right. \\ & \left. -\frac{k^r}{r(j-r)}(1-k^{m-j})\right] + \frac{1}{(1-k^m)^2} \\ & \times \sum_{j=1}^{m-1}\sum_{r=1}^{m-1}\frac{1}{jr}[2(1+k^m)(k^j+k^{m-j}) \\ & -(1+k^m)^2-(k^j+k^{m-j})(k^r+k^{m-r})]. \end{aligned} \tag{70}$$

The dependence of $\bar{t}$ and σ on the number of primary electrons is shown in Figs. 5 and 6. In order to find the asymptotic approximation of $\bar{t}(m)$ and $\sigma(m)$ for large m we use the trick described in Appendix C. We just illustrate the calculation for $\bar{t}(m)$, where we calculate the Z-Transform of $\kappa_1(m)$. We assume that k is small and k^m therefore negligible, which gives

$$\begin{aligned} \mathscr{Z}[\kappa_1(m), m, z] = {} & \sum_{m=0}^{\infty}\frac{\kappa_1(m)}{z^m} \\ & \approx \frac{1}{2\pi i}\oint\left(\sum_{m=0}^{\infty}\frac{1}{z^m z_2^m}\right)J(z_2)\ln M(z_2)\,dz_2 \\ & = \frac{1}{2\pi i}\oint\frac{J(z_2)\ln M(z_2)}{1-z_2 z}\,dz \\ & = \frac{1}{z}J(1/z)\ln M(1/z). \end{aligned} \tag{71}$$

We then replace z by e^s, expand the expression around $s = 0$ and perform the inverse Laplace Transform of this series, which gives

$$\begin{aligned} S\bar{t}(m) \approx {} & \ln n - \ln m + \frac{1+k}{2(1-k)}\frac{1}{m} \\ & + \frac{1+k(10+k)}{12(1-k)^2}\frac{1}{m^2} + O\left(\frac{1}{m^3}\right). \end{aligned} \tag{72}$$

The leading term of this approximation is easily interpreted by the average growth of the avalanche started with m electrons, according to $n(t) = m\exp(St)$. The threshold n is crossed at time $\bar{t} = (\ln n - \ln m)/S$.

Performing the same operation for $\kappa_2(m)$ and using Eq. (64) we find

$$S^2\sigma^2(m) \approx \frac{1+k}{1-k}\frac{1}{m} + \frac{1+k(6+k)}{2(1-k)^2}\frac{1}{m^2} + O\left(\frac{1}{m^3}\right). \tag{73}$$

The first order and second order approximation are superimposed to the exact calculation in Figs. 5 and 6. For large values of m the variance decreases as $1/m$, for small attachment this behavior is also quite accurate for small values of m. The same is true for the average. Finally, we observe that for $k = 0$ we get

$$S\bar{t} = \ln n + \gamma - \sum_{j=1}^{m-1}\frac{1}{j}, \quad S^2\sigma^2 = \frac{\pi^2}{6} - \sum_{j=1}^{m-1}\frac{1}{j^2} \tag{74}$$

as in Ref. [3]. The variance is a quite curious expression: because we know that due to the central limit theorem the variance tends to zero for large primary electrons numbers m, it must hold that $\sum_{j=1}^{\infty}1/j^2 = \pi^2/6$! We have, therefore, solved that famous problem of summing the inverse squares by an RPC detector physics argument! The second order continuous approximation is $S^2\sigma^2 = 1/m + \frac{1}{2}m^2$, which is remarkably accurate even for small values of m.

7. RPC time response function

Finally, we can calculate the RPC time response function for a charged particle passing an RPC gap. We use the Poissonian number of efficient clusters, with average n_0, and the different cluster size distributions as discussed in Section 3. In the following we use only the continuous avalanche approximation.

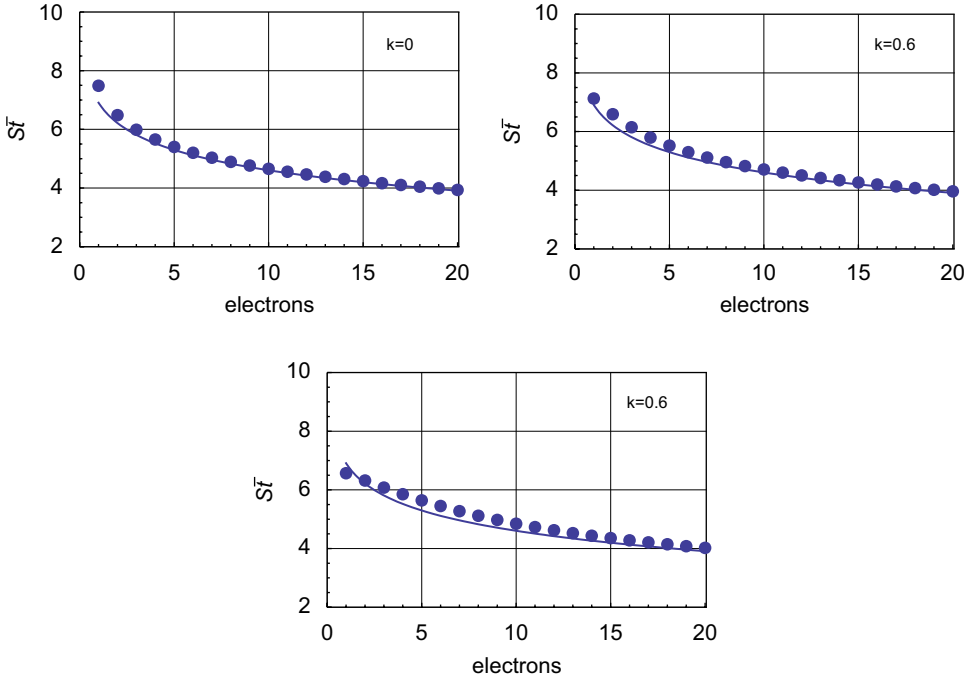

Fig. 5. The figures show the average threshold crossing time for a threshold of $n = 1000$ and $k = 0,\ 0.3,\ 0.6$ versus number of primary electrons. The solid line shows the first term of the approximation from Eq. (72), namely $\ln n - \ln m$.

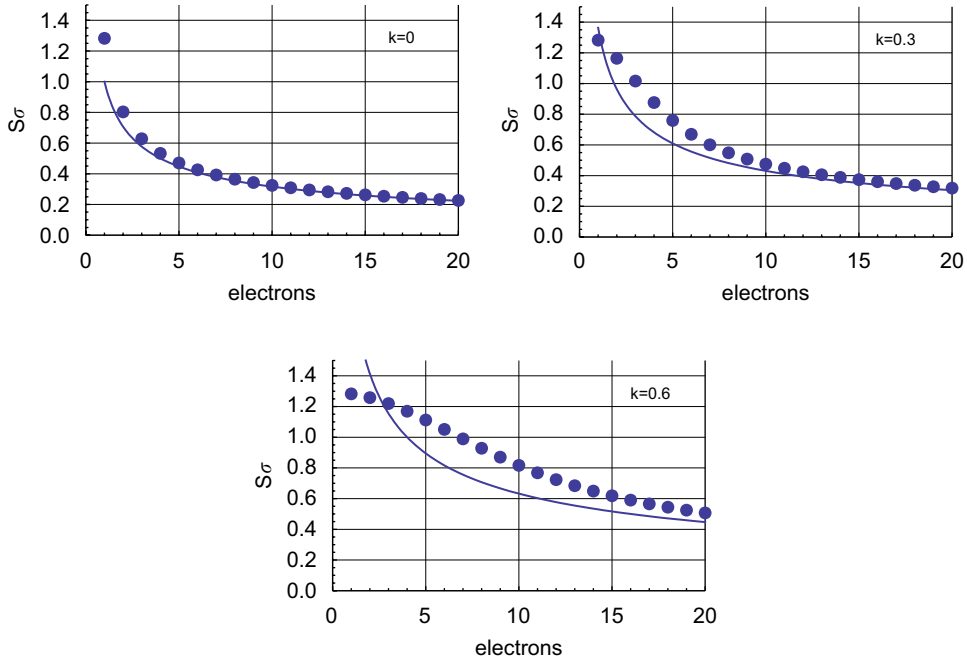

Fig. 6. Variance of the threshold crossing time for $k = 0,\ 0.3,\ 0.6$ versus number of primary electrons. The solid lines show the first order approximation from Eq. (73), namely $(1 + k)/(1 - k)/m$.

7.1. Single electron per cluster, no attachment

We first consider a single electron per cluster in absence of attachment, i.e. $k = 0$ and $F(z) = 1/z$. We have

$$\begin{aligned}\rho(n,t) &= \frac{1}{2\pi i}\oint \frac{e^{n_0/z}-1}{e^{n_0}-1} Sn\,\exp(-St - n(1-z)e^{-St})\,dz \\ &= \frac{Sn\,\exp(-St-ne^{-St})}{e^{n_0}-1}\frac{1}{2\pi i}\oint e^{n_0/z}e^{nze^{-St}}\,dz \\ &= \frac{Sn\,\exp(-St-ne^{-St})}{e^{n_0}-1} \\ &\quad\times\frac{1}{2\pi i}\oint \sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{n_0}{z}\right)^j \sum_{m=0}^{\infty}\frac{1}{m!}(n_0 z e^{-St})^m\,dz \\ &= \frac{Snn_0\,\exp(-St-ne^{-St})}{e^{n_0}-1} \\ &\quad\times\sum_{m=0}^{\infty}\frac{1}{m!(m+1)!}(n_0 n e^{-St})^m \\ &= \frac{Snn_0\,\exp(-St-ne^{-St})}{(e^{n_0}-1)\sqrt{n_0 n e^{-St}}} I_1(2\sqrt{n_0 n e^{-St}})\end{aligned} \tag{75}$$

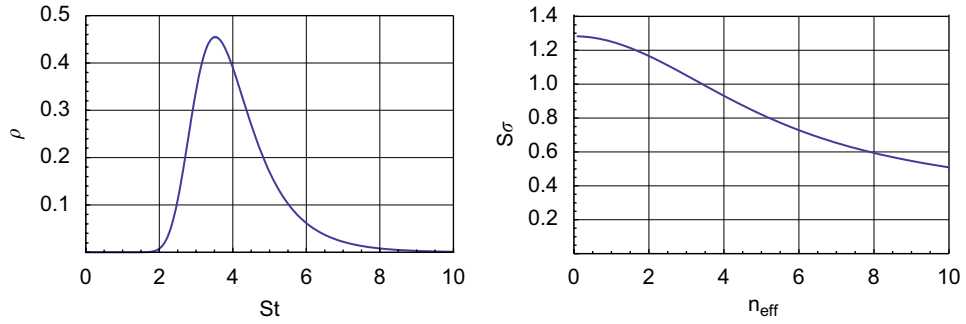

Fig. 7. Universal shape of the time response function valid for a single electron or exponential cluster size distribution in presence or absence of attachment. The number n_{eff} is given by $n_{eff} = n_0(1-k)/(1-kq)$.

where $I_1(x)$ is the modified Bessel Function of first kind. This is the expression derived in Ref. [4]. To find the average threshold crossing time and the variance we use Eqs. (63) and (64). We find

$$\kappa_1 = -\frac{1}{e^{n_0}-1}\sum_{m=1}^{\infty}\frac{n_0^m}{m!}H(m-1)$$
$$\kappa_2 = \frac{1}{e^{n_0}-1}\sum_{m=1}^{\infty}\frac{n_0^m}{m!}\left[H(m-1)^2 - \sum_{j=1}^{m-1}\frac{1}{j^2}\right] \quad (76)$$

which was derived in Ref. [3]. $H(m)$ is the mth Harmonic number defined as $H(m) = \sum_{j=1}^{m} 1/j = \gamma + \psi_0(m+1)$, where the function $\psi_0(x)$ is the Digamma function [14]. The average threshold crossing time $\bar{t}$ and the variance σ are then again given by Eq. (64). The time response function for $n_0 = 5$ and the variance for different numbers of n_0 are shown in Fig. 7.

7.2. Single electron per cluster, attachment

From the attachment transformation property we know that the time response function for $G(z)$ with attachment k is equal to the time response function for $G(z(y))$ and threshold $n' = (1-k)n$ without attachment. For the single electron per cluster we have

$$G(z) = \frac{e^{n_0/z}-1}{e^{n_0}-1}, \quad G(z(y)) = \frac{e^{n_0 k}\, e^{(1-k)n_0/y}-1}{e^{n_0}-1}$$

with $r_G = 0$. This is, however, the same expression where n_0 is replaced by $n_0(1-k)$ (up to normalization factors and a vanishing term $1/(\exp(n_0)-1)$). Since the path of integration in the y-plane is a circle that contains $y = 0$ we have to evaluate the residuals in the same way as above and we have the result. The effect of attachment for a Poissonian primary electron distribution is equal to an effective reduction of the average primary electron number according to $n_{eff} = (1-k)n_0$ [5]. Note that this feature is only due to the property of the exponential, so it is uniquely related to the Poissonian primary electron distribution.

7.3. Exponential cluster size distribution, attachment

The exponential cluster size distribution $F(z) = (1-q)/(z-q)$ gives

$$G(z) = \frac{\exp\left(n_0\frac{1-q}{z-q}\right)-1}{e^{n_0}-1}. \quad (77)$$

This distribution in presence of attachment is equal to $G(z(y))$ without attachment, and we have

$$G(z(y)) = \frac{\exp\left(n_0 k\frac{1-q}{1-kq}\right)\exp\left(n_0\frac{1-k}{1-kq}\frac{1-q'}{y-q'}\right)-1}{e^{n_0}-1}$$

with $q' = q(1-k)/(1-kq)$. From the cluster transformation we know however that $G(z)$ and $G((z-q')/(1-q'))$ give the same time response function, so we only need to use the expression

$$G(z(y)) = \frac{\exp\left(n_0 k\frac{1-q}{1-kq}\right)\exp\left(n_0\frac{1-k}{1-kq}\frac{1}{y}\right)-1}{e^{n_0}-1}$$

in absence of attachment, which gives again the time response function from Eq. (75) just with a different n_{eff}! We conclude: the time response function for an exponential cluster size distribution in presence of attachment is equal to the time response function for a single electron per cluster in absence of attachment with an average number of efficient clusters equal to $n_{eff} = n_0(1-k)/(1-qk)$.

7.4. Realistic cluster size distributions

For the more realistic $1/n^2$ or HEED cluster size distributions we have to evaluate the integrals numerically. For the $1/n^2$ distribution we have

$$\rho(n,t) = \frac{e^{n_0}-1}{e^{n_0}-\exp(n_0\mathrm{Li}_2(k))}\frac{1}{2\pi} \times \int_{-\pi}^{\pi}\frac{\exp(6n_0\mathrm{Li}_2(e^{-i\varphi}/r))/\pi^2-1}{e^{n_0}-1}\frac{(1-k)^2}{(1-kre^{i\varphi})^2} \times Sn\,\exp\left[-St - n\frac{(1-k)(1-re^{i\varphi})}{1-kre^{i\varphi}}e^{-St}\right] r\,e^{i\varphi}\,d\varphi$$

with $1<r<1/k$. The expression can also be evaluated by integrating the real part of the integrand from 0 to π and multiplying the result by 2 (Eqs. (100) and (101)).

For the cluster size distribution from HEED we calculate the cluster size probability distribution p_m up to an electron number of 500 which defines $G(z) = \sum_{m=0}^{500} p_m/z^m$. We then evaluate numerically the expression

$$\rho(n,t) = \frac{r}{2\pi}\int_{-\pi}^{\pi}\frac{G(re^{i\varphi})}{1-G(1/k)}\frac{(1-k)^2}{(1-kre^{i\varphi})^2}Sn \times \exp\left[i\varphi - St - n\frac{(1-k)(1-re^{i\varphi})}{1-kre^{i\varphi}}e^{-St}\right]d\varphi$$

with $0<r<1/k$. The results for the two cluster size models for $k = 0$ together with the result for a single electron per cluster are shown in Fig. 8. We see that the $1/n^2$ and HEED cluster size distributions result in a wider and more symmetric time response function as compared to the un-physical models. Fig. 8 also compares the variance for the different models, and we see that the realistic HEED cluster distribution results in a larger variance compared to the single electron model, but it is still smaller than the one for the naive $1/n^2$ model. In Fig. 9 we see the comparison of the three models for attachment $k = 0.3$ and 0.6, which shows slightly less difference than the case of no attachment. For an

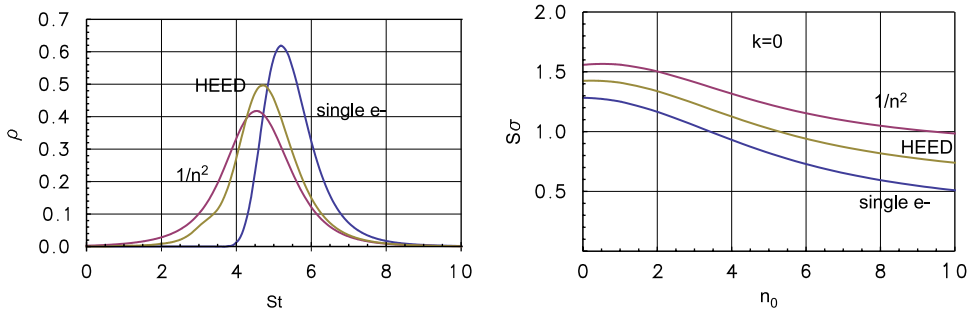

Fig. 8. Time response functions for the single electron, $1/n^2$ and HEED cluster size distributions for $n_0 = 5$ and $k = 0$. For the realistic models the time response function is more symmetric. The tail to the right (later times) is due to avalanche fluctuations while the tail to earlier times is due to frequent events of very large primary electron numbers.

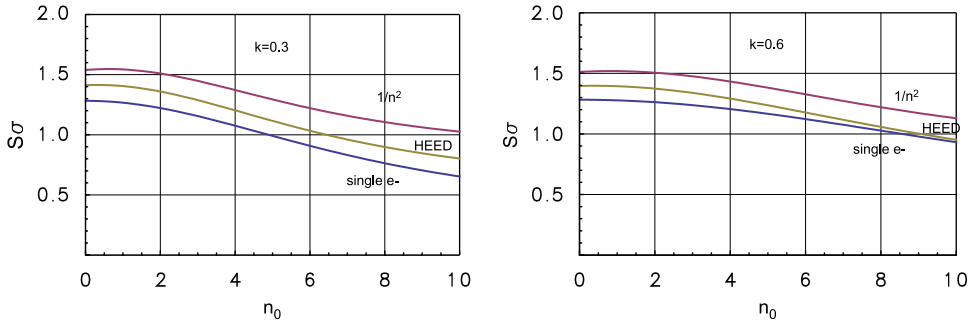

Fig. 9. RPC time resolution versus number of efficient clusters for the three different cluster size distributions for $k = 0.3$ and $k = 0.6$. The differences are smaller compared to the case of no attachment.

typical number of $n_0 = 5$ and attachment $k = 0.2$ the realistic cluster size distribution reduces the time resolution by 10–20% compared to the single electron clusters.

8. Discussion

Effect of cluster sizes different from one: The inclusion of realistic cluster size distributions changes the time response function in two ways. First, the time response function becomes more symmetric, because the finite probability to have a large number of primary electrons produces early threshold crossing times, in contrast to one electron per cluster, where the resulting Poisson distribution does not allow large variations around the average number of primary electrons. Second, the time resolution, or more precisely the variance of the threshold crossing time, is 10–20% larger due to the larger fluctuation of primary electrons. The overall dependence of this variance on the number of efficient clusters is however still not very different from the universal time resolution curve, which is valid for single electron and exponential cluster size distributions, as seen in Fig. 10.

Dependence of σ on n_0: The dependence of the universal time resolution curve on n_0 is discussed in Refs. [3,5]. Using the approximations from Eqs. (72) and (73) we can put the large n_0 behavior into a more accurate form. We split the resolution into the avalanche fluctuation contribution and the primary ionization contribution [3]. In case each electron would produce an average avalanche growing exponentially without fluctuation, the time resolution would be only due to the fluctuation in the primary electron number and the associated shift of the average threshold crossing time $\bar{t}(m)$. If on the other hand the average threshold crossing time would not change, the time resolution would only be due to the avalanche fluctuations $\sigma(m)$ weighted by the different primary electron numbers. If $\bar{t}(m)$ is the average threshold crossing time for m primary electrons (Eq. (69)) and $\sigma^2(m)$ is the variance of the threshold crossing time for m primary electrons (Eq. (70)) then $\overline{t^2}(m) = \bar{t}^2(m) + \sigma^2(m)$ is the second moment of the threshold crossing time. If $g(m)$ is the probability to have m electrons in the gas gap, the first and the second moment of the time response function is

$$\bar{t} = \sum_{m=1}^{\infty} g(m)\bar{t}(m)$$

$$\overline{t^2} = \sum_{m=1}^{\infty} g(m)[\bar{t}^2(m) + \sigma^2(m)]$$

and the variance is $\sigma^2 = \overline{t^2} - \bar{t}^2 = \sigma_1^2 + \sigma_2^2$ where

$$\sigma_1^2 = \sum_{m=1}^{\infty} g(m)\sigma(m)^2$$

$$\sigma_2^2 = \sum_{m=1}^{\infty} g(m)\bar{t}^2(m) - \left(\sum_{m=1}^{\infty} g(m)\bar{t}(m)\right)^2.$$

σ_1 is due to the avalanche fluctuations and σ_2 is due to the change in average threshold crossing time. Using $g(m) = n_0^m/m!(\exp(n_0) - 1)$ the expression becomes equal to the universal time resolution curve calculated in Section 7.1. Evaluation of σ_1 and σ_2 for $k = 0$ as a function of n_0 we arrive at the curves shown in Fig. 11. For small numbers of n_0 we always have one electron and therefore $\sigma_1^2 = \pi^2/6$ and $\sigma_2^2 = 0$. For increasing numbers of n_0 the change in threshold crossing time σ_2 starts to play a role an becomes comparable to σ_1 [5]. This can be quantified in more detail. For large numbers of n_0, both components become equal to $1/\sqrt{n_0}$, which is quite remarkable. For σ_1 this can be understood by the fact that the fluctuation of m around n_0 becomes negligible and from Eq. (73) we know that $\sigma(m)^2 = 1/m$ for large values of m. The fact that also σ_2 becomes equal to $1/\sqrt{n_0}$ becomes clear by the fact that the variance of $g(m)$ is $\sigma_{n_0} = \sqrt{n_0}$ and that for large m we

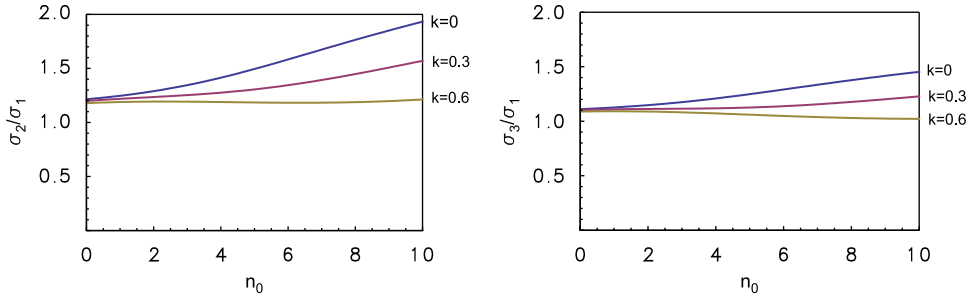

Fig. 10. The first figure shows the time resolution ratio between the $1/n^2$ model and the single electron model. The second figure shows the time resolution ratio between the HEED model and the single electron model.

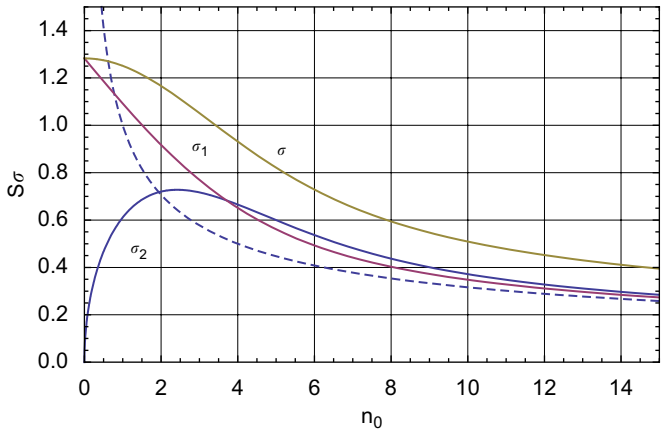

Fig. 11. Universal time resolution σ as a function of n_0 together the components due to avalanche fluctuations σ_1 and average threshold crossing time variations σ_2. The dotted curve is $1/\sqrt{n_0}$.

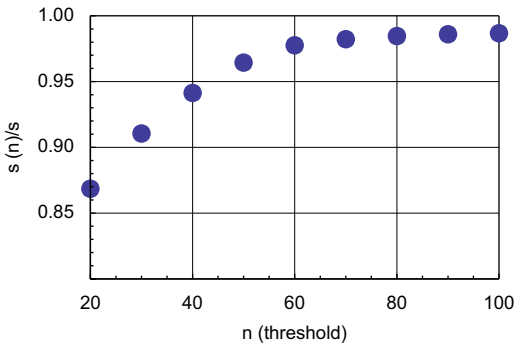

Fig. 12. RPC time resolution versus threshold n for $n_0 = 5$ and $k = 0.2$. The plot shows the ratio between the time resolution for threshold n and the time resolution for very high threshold where σ becomes independent of the threshold.

have $\bar{t}(m) = \ln n - \ln m$. This results in a time variance of

$$\sigma = \sqrt{\left(\frac{d\bar{t}(m)}{dm}|_{m=n_0}\right)^2 \sigma^2_{n_0}} = \frac{1}{\sqrt{n_0}}. \quad (78)$$

The fact that σ_1 and σ_2 become equal is, therefore, tied to the assumption of the Poissonian fluctuation of the primary electron number. The large n_0 approximation of the universal time response function curve is, therefore, given by $\sigma(n_0) = \sqrt{2/n_0}$. For the more realistic models this will not any more be the case, but we have learned that for numbers of $n_0 > \approx 3$ the avalanche fluctuations and primary ionization fluctuations are contribution in very similar amounts to the RPC time resolution.

Threshold effects: For large numbers of electrons in the gas gap, the time resolution is independent of the applied threshold. The question arises whether we can lower the threshold to levels where the continuous approximation does not apply and time variance is smaller. Fig. 12 shows the RPC time resolution for very low thresholds of $n < 100$, calculated for the HEED cluster size distribution for $n_0 = 5$ and $k = 0.2$ with the exact expression for $H(z, n, t)$. We see that already at a threshold of $n = 100$ electrons the time resolution has reached the 'large n' limit. The induced charge $Q(t)$ is on average related to the produced charge $n(t)$ by

$$Q(t) = \frac{E_w}{V_w}\frac{1}{\alpha - \eta} e_0 n(t) \quad (79)$$

with typical values of $E_w/V_w \approx 100\,\mathrm{mm}^{-1}$ and $\alpha - \eta \approx 100\,\mathrm{mm}^{-1}$ the induced charge for $n = 100$ is only of the order of a single elementary charge e_0, so there is no theoretical possibility to have a threshold at such a low level. As illustrated in Ref. [2] it is the very beginning of the avalanche, when only a few multiplications have taken place, that determines the time resolution, so there is no possibility to have a threshold low enough to improve the time resolution.

9. Conclusion

We have derived analytic expressions for the RPC time response function that allow the inclusion of realistic cluster size distributions. Analytic expressions for the average threshold crossing time and the variance are given as well. Using a $1/n^2$ or more accurate HEED cluster size distribution we find that the time response function is more symmetric compared to the one where a single electron per cluster is assumed. The time resolution is not any more a universal function of an effective number of clusters n_{eff}. The difference to the single electron model are of the order of 10–20%.

The independence of the RPC time resolution from the applied threshold is a general feature of the avalanche fluctuations and is independent of the specific primary electron distribution. At very low thresholds n the continuous approximation of the avalanche fluctuations is not appropriate. Using the exact expression for the avalanche fluctuation we see that only at thresholds of $n < 50$ electrons the time resolution improves, and in practice such low thresholds are unrealistic.

The two contributions to the RPC time resolution, namely the change in average threshold crossing time for different primary electron numbers and the actual avalanche fluctuation become comparable for $n_{eff} \approx 3$ and become equivalent for larger cluster numbers.

Appendix A. Z-Transform

The Z-Transform of a sequence $f(n)$ is defined by

$$\mathscr{Z}[f(n),n,z]=F(z)=\sum_{n=0}^{\infty}\frac{f(n)}{z^n} \tag{80}$$

which converges for $|z|>r_F$ and we call r_F the radius of convergence for $F(z)$. The Z-Transform of a two dimensional sequence $f(m,n)$ is defined by

$$\begin{aligned}F(z_1,z_2)&=\mathscr{Z}[f(n,m),n,z_1,m,z_2]\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{f(n,m)}{z_1^n z_2^m}.\end{aligned} \tag{81}$$

There are several ways to invert the Z-Transform

$$f(n)=\mathscr{Z}^{-1}[F(z),z,n] \tag{82}$$

$$=\frac{1}{2\pi \mathrm{i}}\oint F(z)z^{n-1}\,\mathrm{d}z \tag{83}$$

$$=\frac{r^n}{2\pi}\int_{-\pi}^{\pi}F(r\,\mathrm{e}^{\mathrm{i}\phi})\,\mathrm{e}^{\mathrm{i}n\phi}\,\mathrm{d}\phi \tag{84}$$

$$=\frac{1}{n!}\left[\frac{d^nF(1/z)}{\mathrm{d}z^n}\right]_{z=0} \tag{85}$$

$$=\lim_{z\to\infty}z^n\left(F(z)-\sum_{m=0}^{n-1}\frac{f(m)}{z^m}\right). \tag{86}$$

The last formula can be used for iterative inversion. The following relations hold:

$$\mathscr{Z}\left[\sum_{m=0}^{n}f(m),n,z\right]=\frac{z}{z-1}F(z) \tag{87}$$

$$\mathscr{Z}\left[\sum_{m=0}^{n}g(n-m)f(m),n,z\right]=F(z)G(z) \tag{88}$$

$$\mathscr{Z}[f(n)g(n),n,z]=\frac{1}{2\pi \mathrm{i}}\oint F(\xi)G\left(\frac{z}{\xi}\right)\frac{\mathrm{d}\xi}{\xi} \tag{89}$$

$$\lim_{n\to\infty}f(n)=\lim_{z\to 1+}(z-1)F(z) \tag{90}$$

$$\sum_{n=0}^{\infty}f(n)=1\quad\rightarrow\quad\lim_{z\to 1}zF(z)=1 \tag{91}$$

$$\mathscr{Z}[nf(n),n,z]=-z\frac{\mathrm{d}F(z)}{\mathrm{d}z} \tag{92}$$

Appendix B. Laplace Transform

The Laplace Transform of a continuous function $f(t)$ is given by

$$F(s)=\mathscr{L}[f(t),t,s]=\int_0^{\infty}f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t \tag{93}$$

and the inverse is given by

$$f(t)=\mathscr{L}^{-1}[F(s),s,t]=\frac{1}{2\pi \mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+i\infty}F(s)\,\mathrm{e}^{st}\,\mathrm{d}s.$$

We make use of the following relations:

$$\mathscr{L}\left[\int_0^t f(t')\,\mathrm{d}t',t,s\right]=\frac{1}{s}F(s) \tag{94}$$

$$\lim_{n\to\infty}f(n)=\lim_{s\to 0}sF(s). \tag{95}$$

Appendix C. Continuous approximation of discrete series

We want to find a continuous 'large n' approximation for a discrete series $f(n)$. We first calculate the Z-Transform $F(z)$ of $f(n)$. In case the difference between $f(n)$ and $f(n+1)$ becomes small, the Laplace Transform of the continuous function $f(n)$ becomes equal to the Z-Transform when replacing z by e^s

$$\int_0^{\infty}f(n)\,\mathrm{e}^{-sn}\,\mathrm{d}n\approx\sum_{n=0}^{\infty}f(n)\,\mathrm{e}^{-sn}=\sum_{n=0}^{\infty}\frac{f(n)}{z^n}.$$

Since the value of $f(n)$ for $n\to\infty$ is found by investigation the limit of the Laplace Transform for $s\to 0$ (Eq. (95)) the continuous approximation of $f(n)$ for large n is found by expanding $F(\mathrm{e}^s)$ for small s and performing the inverse Laplace Transform of the expression. As an example we investigate the so called harmonic numbers

$$f(n)=\sum_{m=1}^{n}\frac{1}{m}$$

$$F(z)=\mathscr{Z}[f(n),n,z]=\frac{z}{z-1}\ln\frac{z}{z-1}.$$

Expanding $F(\mathrm{e}^s)$ for small s gives

$$F(\mathrm{e}^s)\approx\frac{1}{2}-\frac{\ln s}{s}-\frac{1}{2}\ln s+O(s). \tag{96}$$

The inverse Laplace Transform of this expression is

$$=\gamma+\ln n+\frac{1}{2n} \tag{97}$$

which is the well know large n approximation of the harmonic numbers [15].

Appendix D. Complex integrals

The contour integral $1/2\pi\mathrm{i}\oint F(z)\,\mathrm{d}z$ of a complex function around a closed path is equal to the sum of residuals inside the contour. The residual at a pole z_0 is equal to the coefficient c_{-1} of the Laurent series around $z=z_0$. In case $F(z)$ has an mth order pole at $z=z_0$ then the residual can be calculated by

$$c_{-1}=\frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{\mathrm{d}z^{m-1}}[(z-z_0)^mF(z)]. \tag{98}$$

In case the path of integration is a circle of radius r around the origin we can write the integral as

$$\frac{1}{2\pi \mathrm{i}}\oint F(z)\,\mathrm{d}z=\frac{1}{2\pi}\int_{-\pi}^{\pi}F(r\,\mathrm{e}^{\mathrm{i}\phi})r\,\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi. \tag{99}$$

In case the function $F(z)$ has the property that $F(\bar{z})=\overline{F(z)}$ this integral is real and equal to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}F(r\,\mathrm{e}^{\mathrm{i}\phi})r\,\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi=\mathrm{Re}\left[\frac{1}{\pi}\int_0^{\pi}F(r\,\mathrm{e}^{\mathrm{i}\phi})r\,\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi\right]. \tag{100}$$

If $F(z)$ is the Z-Transform of a real series it has this property

$$\begin{aligned}&F(z)=\sum\frac{a_n}{z^n}\quad a_n\in R\quad\rightarrow\\&F(\bar{z})=\sum\frac{a_n}{\bar{z}^n}=\sum\frac{a_n}{\overline{z^n}}=\overline{F(z)}.\end{aligned} \tag{101}$$

Appendix E. Solution of Legler's avalanche model

Writing $P(x,z)=X(x)Z(z)$ we can separate Eq. (23)

$$\frac{X'}{X}=\frac{Z'}{Z}((\alpha+\eta)z-\alpha-\eta z^2)=\lambda. \tag{102}$$

The solutions of the two equations are

$$X(x) = \mathrm{e}^{\lambda x}, \quad Z(z) = \left(\frac{z-1}{1-kz}\right)^{\lambda/(\alpha-\eta)} \tag{103}$$

and the general solution is, therefore,

$$P(x,z) = \int a(\lambda)\left(\mathrm{e}^{(\alpha-\eta)x}\frac{z-1}{1-kz}\right)^{\lambda/(\alpha-\eta)} \mathrm{d}\lambda \tag{104}$$

with an arbitrary function $a(\lambda)$. The condition to have m electrons at $x = 0$ results in the relation

$$P(0,z) = \frac{1}{z^m} = \int a(\lambda)\left(\frac{z-1}{1-kz}\right)^{\lambda/(\alpha-\eta)} \mathrm{d}\lambda \tag{105}$$

which defines $a(\lambda)$. Denoting the expression $(z-1)/(1-kz)$ by y, the relation reads as

$$\int a(\lambda) y^{\lambda/(\alpha-\eta)}\, \mathrm{d}\lambda = \left(\frac{1+ky}{1+y}\right)^m \tag{106}$$

so by inserting the expression in the brackets of Eq. (104) for y we have the final solution

$$P(z,x) = \left(\frac{\eta(z-1)\,\mathrm{e}^{(\alpha-\eta)x} - z\eta + \alpha}{\alpha(z-1)\,\mathrm{e}^{(\alpha-\eta)x} - z\eta + \alpha}\right)^m. \tag{107}$$

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